On Links between Concept Lattices and Related Complexity Problems

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Abstract. Several notions of links between contexts – intensionally related concepts, shared intents, and bonds, as well as interrelations thereof – are considered. Algorithmic complexity of the problems related to respective closure operators are studied. The expression of bonds in terms of shared intents is given.

1 Introduction

In many applications one has to deal with data about a system changing in time. To analyze this dynamics one has to track similarities between different states of the system. The change of data makes the knowledge about the system change too. When one uses Formal Concept Analysis as the underlying mathematical model to describe this kind of dynamics, one needs to have tools for finding "similar" concepts in two concept lattices and "links" between similar concepts in them. For example, in [8] the study of dynamics of a scientific community is based on finding similar concepts of contexts that represent same community at different time. Similarity of concepts play important role in an earlier model of a network of concepts based on *multicontexts* [11]. In this paper we consider several notions that reflect links between concepts lattices, such as intensionally related concepts, shared intents, and bonds. We study algorithmic complexity of the problems related to computing closures and maximally related concepts. We also study the relation between shared intents and bonds.

The paper is organized as follows. In Section 2, we introduce basic definitions and discuss intensionally related concepts. In Section 3 we study shared intents of two contexts and study some important complexity problems related to shared intents. We also show that shared intents are related to bonds between contexts.

2 Intentionally Related Concepts

First we recall some basic notions of Formal Concept Analysis (FCA) [10,2].

Let G and M be sets, called the set of objects and the set of attributes, respectively. Let I be a relation $I \subseteq G \times M$ between objects and attributes: for $g \in G$, $m \in M$, gIm holds iff the object g has the attribute m. The triple

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$K = (G, M, I)$ is called a *(formal) context*. Formal contexts are naturally represented by cross tables, where a cross for a pair (g, m) means that this pair belongs to the relation I. If $A \subseteq G$, $B \subseteq M$ are arbitrary subsets, then the *Galois connection* is given by the following *derivation operators*:

$$A' := \{m \in M \mid gIm \text{ for all } g \in A\},$$
$$B' := \{g \in G \mid gIm \text{ for all } m \in B\}.$$

The pair (A, B), where $A \subseteq G$, $B \subseteq M$, $A' = B$, and $B' = A$ is called a *(formal) concept (of the context K)* with *extent* A and *intent* B. For $g \in G$ and $m \in M$ the sets $\{g\}'$ and $\{m\}'$ are called *object intent* and *attribute extent*, respectively. The set of attributes B *is implied by the set of attributes* D, or an *implication* $D \to B$ holds, if all objects from G that have all attributes from the set D also have all attributes from the set B, i.e., $D' \subseteq B'$.

The operation $(\cdot)''$ is a closure operator [2], i.e., it is idempotent ($X'''' = X''$), extensive ($X \subseteq X''$), and monotone ($X \subseteq Y \Rightarrow X'' \subseteq Y''$). Sets $A \subseteq G$, $B \subseteq M$ are called *closed* if $A'' = A$ and $B'' = B$. Obviously, extents and intents are closed sets. Since the closed sets form a closure system or a Moore space [1], the set of all formal concepts of the context K forms a lattice, called a *concept lattice* and usually denoted by $\underline{\mathfrak{B}}(K)$ in FCA literature.

Let $K_1 = (G_1, M, I_1), K_2 = (G_2, M, I_2), \ldots, K_r = (G_r, M, I_r)$ be contexts with common attribute set M. Denote by $(\cdot)^i$ the derivation operator in context K_i. A tuple of r concepts $(A_1, B_1), (A_2, B_2), \ldots, (A_r, B_r))$ of corresponding contexts $K_1, K_2, \ldots, K_r$ are called *intentionally related* [8] if

$$(\bigcap_{1 \leq i \leq r} A_i)^{11} = A_1$$
$$(\bigcap_{1 \leq i \leq r} A_i)^{22} = A_2$$
$$\ldots$$
$$(\bigcap_{1 \leq i \leq r} A_i)^{rr} = A_r$$

So any intentionally related concepts are uniquely defined by the set $\bigcap_{1 \leq i \leq r} A_i$.

Consider an operator $(\cdot)^*$, defined as $X^* = X^{11} \cap X^{22} \cap \ldots \cap X^{rr}$ for $X \subseteq M$.

Proposition 1. *Let $K_1 = (G_1, M, I_1), K_2 = (G_2, M, I_2), \ldots, K_r = (G_r, M, I_r)$ be contexts with common attribute set M. Then the operator $(\cdot)^*$ has the following properties:*
(1) $(X^*)^{ii} = X^{ii}$, for any $X \subseteq M$ and $1 \leq i \leq r$.
(2) $(\cdot)^*$ is a closure operator.

Proof. (1) Indeed, $(X^*)^{ii} = (\bigcap_{1 \leq j \leq r} X^{jj})^{ii}$, since $X \subseteq X^{jj}$ for any $1 \leq j \leq r$ it follows that $X \subseteq \bigcap_{1 \leq j \leq r} X^{jj}$ and hence $X^{ii} \subseteq (X^*)^{ii}$. On the other hand $\bigcap_{1 \leq j \leq r} X^{jj} \subseteq X^{ii}$, therefore $(X^*)^{ii} \subseteq X^{ii}$.

(2) It is not hard to check that this operator is a closure operator:
1. $X \subseteq Y \Rightarrow X^{ii} \subseteq Y^{ii}$, for $1 \leq \mathrm{i} \leq \mathrm{r} \Rightarrow X^* \subseteq Y^*$ (monotony)
2. $X \subseteq X^*$(was proved above) (extensity)
3. $X^{**} = \bigcap_{1 \leq j \leq r} (X^*)^{jj} = \bigcap_{1 \leq j \leq r} X^{jj} = X^*$ (idempotency)

□

The situation is easily extended to the case where contexts $K_1, K_2, \ldots, K_r$ have different sets of attributes $M_1, M_2, \ldots, M_r$. One defines $M := \bigcup_{i=1}^{r} M_i$ and proceeds like above.

Having the closure operator $(\cdot)^*$, one can compute all intentionally related concepts by standard algorithms (Norris, Next Closure, Close-by-One, etc.).

3 Concepts with Shared Intents

As in the previous section, let $K_1 = (G_1, M, I_1), K_2 = (G_2, M, I_2), \ldots, K_r = (G_r, M, I_r)$ be contexts with common attribute set M, $(\cdot)^i$ denotes the derivation operator in K_i for $1 \leq i \leq k$. A set $A \subseteq M$ is called *shared intent* for contexts $K_1, K_2, \ldots, K_r$ if it is an intent for every context K_i, i.e. $A^{ii} = A$.

Since for any context the set of all its intents generates a closure system, the set of all shared intents also generates a closure system. Let us denote the corresponding closure operator by $(\cdot)^{\mathcal{S}}$.

Theorem 2. *The problem*

INPUT Formal contexts $K_1 = (G_1, M, I_1), K_2 = (G_2, M, I_2), \ldots, K_r = (G_r, M, I_r)$ with common attribute set M, and a set $X \subseteq M$.

OUTPUT The closure $X^{\mathcal{S}}$ of X.

can be solved in $O(|M| \sum_{1 \leq i \leq r} |G_i|)$ time.

Proof. Consider sets $S_i = \{g' \mid g \in G_i, X \subseteq g'\} \cup \{M\}$, $1 \leq i \leq r$. Denote by $\bigcap S_i = \bigcap_{A \in S_i} A$. We will keep invariant that for every $1 \leq i \leq r$, any shared intent that contains X can be obtained by intersection of some elements of S_i.

Suppose that there exists an attribute $m \in M$ such that for some $1 \leq i \leq r$ one has $m \in \bigcap S_i$. Then, since every shared intent that contains X can be obtained by intersection of some elements of S_i, every shared intent that contains X have to contain m. Hence, if for some $1 \leq j \leq r$ there is an element $A \in S_j$ which does not contain m, we can update S_j by removing this element while keeping the invariant. When no such removal can be done, we try to find another element $m \in M$ that is contained in all elements of some S_i, and so on. Since M is finite, at some step there is no such $m \in M$. This means that any element $m \in M$ either belongs to every element of any $S_i, 1 \leq i \leq r$ or it does not belong to some element of S_i for every $1 \leq i \leq r$. Hence $\bigcap S_1 = \bigcap S_2 = \ldots = \bigcap S_r$, $X \subseteq \bigcap S_1$ and $\bigcap S_1$ is contained in any shared intent that contains X i.e. $\bigcap S_1 = X^{\mathcal{S}}$.

```
GetClosure(X)
answer ← X
for i ← 1 to r
      do remove all rows from I[i] that do not contain X
for i ← 1 to r
      do for m ← 1 to |M|
             do if not answer[m]
                  then if I[i][j][m] = TRUE for all 1 ≤ j ≤ |G_i|
                         then answer[m] ← TRUE
                              push m in shared-attributes
                         else for each 1 ≤ j ≤ |G_i| such that not I[i][j][m]
                                 do push (j, i) in not-in[m]
                                    counter[i][m] ← counter[i][m] + 1
while shared-attributes not empty
      do pop m from shared-attributes
         while not-in[m] not empty
             do pop (object-index, context-index) from not-in[m]
                for i ← 1 to |M|
                    do if not I[i][context-index][object-index]
                         then counter[context-index][i] ←
                              ← counter[context-index][i] −1
                              if counter[context-index][i] ≤ 0
                              and not answer[i]
                                 then answer[i] = TRUE
                                      push i in shared-attributes
return answer
```

Fig. 1. Pseudocode of the algorithm for computing $(\cdot)^{\mathcal{S}}$

Here $I[i]$ is a binary table representation of relation I_i, *counter*$[i][m]$ is the number of objects g of G_i for which gI_im does not hold, *shared-attributes* and *not-in*$[m]$ can be implemented as *stacks* or as any other data structure, with operations "pop" any object from it and "push" any object in it in $O(1)$ time. □

The problem of finding a maximal cardinality closed set wrt. $(\cdot)'$ and $(\cdot)^*$ is trivially polynomial: one just checks all object intents and finds the largest one. In contrast to that, a similar problem of finding a maximal size shared intent different from M (in practical data analysis this may correspond to the largest similarity of two contexts) for operator $(\cdot)^{\mathcal{S}}$ is NP-complete, as shown in the following

Proposition 3. *The problem*

INPUT Two formal contexts $K_1 = (G_1, M, I_1), K_2 = (G_2, M, I_2)$, and integer $0 \leq k \leq |M|$.

QUESTION Does there exist a set X such that $X = X^{\mathcal{S}}$, $X \subset M$ and $|X| \geq k$?

is NP-complete.

Proof. By Theorem 2 $X = X^{\mathcal{S}}$ can be checked in polynomial time, so the problem is in NP. For proving NP-hardness we reduce a well-known NP-complete problem – *minimal set cover* (MSC) to this one. The MSC problem is formulated as follows [4]:

INPUT Finite set S, some set of its subsets $\mathcal{P} := \{S_1, S_2, \ldots, S_m\}$, $S_i \subseteq S$, and integer k.

QUESTION Does there exist a subset $\mathcal{T} \subseteq \mathcal{P}$ such that $\bigcup_{X \in \mathcal{T}} X = S$ and $|\mathcal{T}| \geq k$?

Consider an arbitrarily finite set $S = \{s_1, s_2, \ldots, s_n\}$, a set of its subsets $\mathcal{P} := \{S_1, S_2, \ldots, S_m\}$, $S_i \subseteq S$ for all $1 \leq i \leq m$, and an integer $0 \leq k \leq |\mathcal{P}|$. Let us define sets $M = \{m_1, m_2, \ldots, m_{|S|}, m_{|S|+1}, m_{|S|+2}, \ldots, m_{|S|+|\mathcal{P}|}\}$ and $G_1 = \{g_1^1, g_2^1, \ldots, g_{|\mathcal{P}|}^1\}$. Now construct context $K_1 = (G_1, M, I_1)$, where I_1 is defined as follows: $g_i^1 I_1 m_j$ for $j \leq |S|$ iff $s_j \notin S_i$ and $g_i^1 I_1 m_j$ for $j > |S|$ iff $j - |S| \neq i$. Then covers of S are in one-to-one correspondence with intents of K_1 that do not contain any $m_i \in M$ for $i \leq |S|$. Moreover, for any set cover of size N, the corresponding intent has size $|\mathcal{P}| - N$.

Now let us construct context $K_2 = (G_2, M, I_2)$, where $G_2 = \{g_1^2, g_2^2, \ldots, g_{|\mathcal{P}|}^2\}$, I_2 is defined as follows: for g_i^2, where $1 \leq i \leq |\mathcal{P}|$, $g_i' = M \setminus (S \cup \{m_{|S|+i}\})$. Obviously, the set of all intents of this context is exactly the set of all subsets M that are disjoint with S. Hence there is one-to-one correspondence between the set of shared intents of K_1 and K_2 excluding M, and set of all set covers. Moreover the minimal covers correspond to maximal shared intents, and maximal shared intents correspond to minimal covers. The reduction is proved, its polynomiality is obvious. □

It is interesting to know how large can be the context for the set of all shared intents? The answer is given by the following proposition:

Proposition 4. *There exist two contexts K_1 and K_2 such that the set of maximal (by inclusion) their shared intents is exponential in size of K_1 and K_2.*

Proof. Consider finite set $S = \{s_1, s_2, \ldots, s_{3n}\}$, and set of its subsets $\mathcal{P} = \bigcup_{0 \leq i \leq n-1} \{\{s_{3i+1}, s_{3i+2}\}, \{s_{3i+1}, s_{3i+3}\}, \{s_{3i+2}, s_{3i+3}\}\}$. There are 3^n minimal covers of S, since for any $0 \leq i \leq n-1$ the subset $\{s_{3i+1}, s_{3i+2}, s_{3i+3}\}$ can be covered only in three ways, using exactly 2 elements from $\mathcal{P}$. So, if we construct contexts K_1 and K_2 like in Proposition 3 according to such S and $\mathcal{P}$, then there are 3^n maximal (by cardinality, and hence by inclusion) shared intents of K_1 and K_2. □

Corollary. There exist two contexts K_1 and K_2 such that the minimal number of objects in the context representing $(\cdot)^{\mathcal{S}}$ is exponential in sizes of K_1 and K_2.

3.1 Bonds as Shared Intents

In [2] the following definition of a bond was given

Definition 5. *Let $K_1 = (G_1, M_1, I_1)$ and $K_2 = (G_2, M_2, I_2)$ be contexts. A relation $I \subseteq G_1 \times M_2$ is called a* bond *from $K_1 = (G_1, M_1, I_1)$ to $K_2 = (G_2, M_2, I_2)$ if m^I is extent of K_1 for any $m \in M_2$, and g^I is intent of K_2 for any $g \in G_1$.*

Recall some definitions from [2]. The *direct product* of contexts $K_1 = (G_1, M_1, I_1)$ and $K_2 = (G_2, M_2, I_2)$ is given by

$$K_1 \times K_2 := (G_1 \times G_2, M_1 \times M_2, \nabla)$$

$$\text{with } (g_1, g_2)\nabla(m_1, m_2) \Leftrightarrow g_1 I_1 m_1 \text{ or } g_2 I_2 m_2.$$

Contranominal scale N_S^c is the context $(S, S, \neq)$

Proposition 6. *A relation $B \subseteq G_1 \times M_2$ is a bond from context $K_1 = (G_1, M_1, I_1)$ to context $K_2 = (G_2, M_2, I_2)$ iff B is a shared intent of contexts $N_{G_1}^c \times K_2$ and $(K_1 \times N_{M_2}^c)^d$.*

Proof. Let $B \subseteq G_1 \times M_2$ be a shared intent of $N_{G_1}^c \times K_2 = (G_1 \times G_2, G_1 \times M_2, \nabla^2)$ and $(K_1 \times N_{M_2}^c)^d = (G_1 \times M_2, M_1 \times M_2, \nabla^1)^d$. Since B is an intent of $N_{G_1}^c \times K_2$, one has

$$B = \bigcap_{(u,h)\in B^{\nabla^2}} \{(g, m) \mid (u, h)\nabla^2(g, m)\}$$

$$= \bigcap_u \bigcap_{h\in H(u)} (\{(u, m) \mid hI_2 m\} \cup \{(g, m) \mid g \neq u\}),$$

where $(g, m) \in G_1 \times M_2$, $\bigcap_u$ is taken over all u such that $(u, h) \in B^{\nabla^2}$ for some h, and $H(u) = \{h \mid (u, h) \in B^{\nabla^2}\}$. Hence if $g = u$ for some u taking part in intersection $\bigcap_u$ we have $g^B = \bigcap_{h\in H(u)} h^{I_2}$, i.e. g^B is closed in K_2, and if $g \neq u$ for all u taking part in $\bigcap_u$ then $g^B = M_2$. Similarly, since B is intent of $(K_1 \times N_{M_2}^c)^d$, we can prove that m^B is closed in context K_1 for any $m \in M_2$.

Let $B \subseteq G_1 \times M_2$ be a bond from context K_1 to context K_2. Then u^B is closed in K_2 for any $u \in G_1$. Denote $H(u) = (u^B)^{I_2}$, then $u^B = \bigcap_{h\in H(u)} h^{I_2}$. Consider

$$D = \bigcap_u \bigcap_{h\in H(u)} (\{(u, m) \mid hI_2 m\} \cup \{(g, m) \mid g \neq u\}).$$

Above we showed that this set is an intent of both $N_{G_1}^c \times K_2$ and $(K_1 \times N_{M_2}^c)^d$, i.e., a shared intent of these contexts. Then $g^D = g^B$ for any $g \in G_1$ and $m^D = m^B$ for any $m \in M_2$, hence $D = B$, and B is a shared intent of $N_{G_1}^c \times K_2$ and $(K_1 \times N_{M_2}^c)^d$. □

Corollary. The closure operator for bonds of contexts $K_1 = (G_1, M_1, I_1)$ and $K_2 = (G_2, M_2, I_2)$ can be computed in $O((|G_1| \cdot |G_2| + |M_1| \cdot |M_2|) \cdot |M_2||G_1|)$ time.

Proof. Apply Proposition 6 and Theorem 2.

Conclusion

We considered several definitions that relate concepts of different contexts, such as intentionally related concepts, shared intents, and bonds. The types of links between concepts of different contexts are important for the study of context dynamics. Algorithmic complexity of problems related to corresponding closure operators was studied. We showed that bonds may be described in terms of shared intents. As further research we would like to find an answer to the question raised in [9] on whether bonds have natural concise context representation.

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